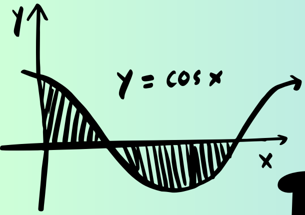
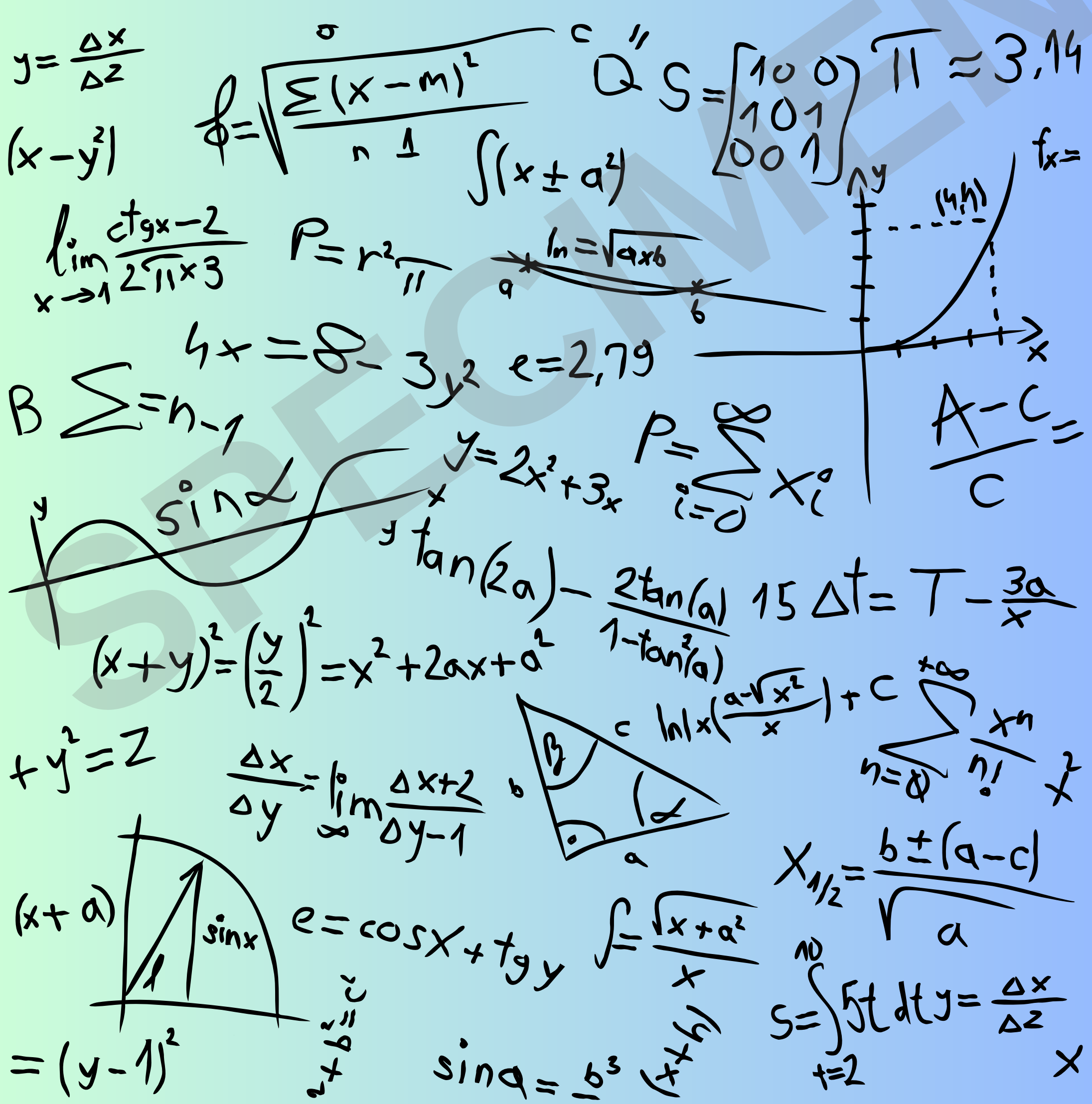


The Art of Integration

From beginner to advanced integration



BY AARAV GANDEWAR

THE ART OF INTEGRATION

A Masterpiece on Integral Techniques

By Aarav Gandewar

"To know the laws governing a phenomenon is not enough; one must integrate them to see the whole."

— Henri Poincaré

THE ART OF INTEGRATION

By Aarav Gandewar

First Edition (2025)

Disclaimer. This book is intended for educational purposes. The author has made every effort to ensure the accuracy of the content, but assumes no responsibility for any errors.

Preface

This book is designed to guide the reader from the foundational principles of integration to some of the most advanced techniques encountered in problem solving and mathematical competitions.

A solid understanding of limits, derivatives, and pre-calculus techniques is recommended, as these concepts are frequently used throughout the book. The focus is not merely on obtaining results, but on understanding why certain methods work and how different ideas connect across seemingly unrelated problems.

Each chapter introduces key theorems and techniques with clear explanations, followed by carefully selected worked examples. Practice problems are included and complete solutions are provided at the end of the book to support independent study.

Several chapters are specifically written with mathematical competitions in mind, offering a strong preparation base for contests centred on integration, such as integration bees. At the same time, the book aims to serve as a long-term reference for anyone who appreciates the art and structure of integral calculus.

This book is intended to be self-contained and does not assume familiarity with any particular text.

Contents

Chapter 1

Foundation of Integration

1.1 Definition

Integration can be viewed as an *inverse operation* to differentiation. Let $f(x)$ be a function such that it is the derivative of another function $F(x)$:

$$\frac{dF(x)}{dx} = f(x)$$

Then, we can *integrate $f(x)$ with respect to x*:

$$\int f(x)\,dx = F(x) + C$$

where C is an arbitrary constant and $F(x)$ is called the **antiderivative** of $f(x)$.

Example 1.
Evaluate

$$\int 2x\,dx$$

Solution.
Since

$$\frac{d}{dx}(x^2) = 2x$$

Differentiating x^2, gives us $2x$. Hence, the antiderivative of $2x$ is $\boxed{x^2 + C}$.

To understand why the constant C appears, note that differentiating $x^2 + C$ gives

$$\frac{d}{dx}(x^2 + C) = 2x + 0 = 2x.$$

Since all constants vanish upon differentiation, there are infinitely many antiderivatives differing only by a constant.

Thus, the general form of an indefinite integral is:

$$\int f(x)\,dx = F(x) + C.$$

Components of an integral:

- $\int$ — the integral symbol
- $f(x)$ — the integrand
- x — the variable of integration
- $F(x)$ — the antiderivative
- C — an arbitrary constant

1.2 Basic Rules of Integration

1. $\displaystyle\int a\,dx = ax + C \qquad (a \text{ is a constant})$
2. $\displaystyle\int x\,dx = \frac{x^2}{2} + C$
3. $\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \qquad (n \neq -1)$
4. $\displaystyle\int af(x)\,dx = a\int f(x)\;dx$
5. $\displaystyle\int [f(x) \pm g(x)]\;dx = \int f(x)\;dx \pm \int g(x)\;dx$

Example 2.
Evaluate

$$\int (9x^4 + 2x^2)\ dx$$

Solution.

$$\begin{aligned}\int (9x^4 + 2x^2)\ dx &= 9\int x^4\ dx\ +\ 2\int x^2\ dx \\ &= 9 \cdot \frac{x^5}{5} + C_1 + 2 \cdot \frac{x^3}{3} + C_2 \\ &= \boxed{\frac{9x^5}{5} + \frac{2x^3}{3} + C}\end{aligned}$$

Recall that the integration constant is *arbitrary*; therefore, the sum of two constants is simply another constant. Thus, $C_1 + C_2 = C$.

Now, you can apply the basic rules of integration. Try these exercises!

Practice 3.
Evaluate:

1. $\int 2x^3\, dx$
2. $\int 8x^9 - 1\, dx$
3. $\int e^x\, dx$
4. $\int \underbrace{x + x + x + \cdots + x}_{\text{2025 times}}\ dx$

1.3 Exponential and Logarithmic Integrals

Now, I shall discuss two fundamental families of integrals you must know

Theorem 1.3.1. *For all real* $x \neq 0$,

$$\int \frac{1}{x}\, dx = \ln|x| + C$$

Theorem 1.3.2. *For any real constant $a > 0$, $a \neq 1$, the integral of a^x with respect to x is*

$$\int a^x \, dx = \frac{a^x}{\ln a} + C.$$

Here is a proof of the above theorem:

Proof. Let $y = a^x$. Then,

$$\ln y = \ln(a^x) = x \ln a.$$

Then, we differentiate both sides with respect to x

$$\frac{1}{y} \frac{dy}{dx} = \ln a.$$

$$\frac{dy}{dx} = y \ln a = a^x \ln a.$$

$$a^x = \frac{1}{\ln a} \frac{dy}{dx}.$$

Now, we integrate both sides with respect to x,

$$\int a^x \, dx = \frac{1}{\ln a} \int \frac{dy}{dx} \, dx = \frac{1}{\ln a} y + C.$$

Finally, substituting $y = a^x$ back,

$$\boxed{\int a^x \, dx = \frac{a^x}{\ln a} + C.}$$

□

The examples below shall be a direct application of the above theorems.

Example 4.
Evaluate

$$\int \left(\frac{10x^{2025}}{x^{2026}} + 2025^x \right) dx$$

Solution.

$$\begin{aligned} \int \left(\frac{10x^{2025}}{x^{2026}} + 2025^x \right) dx &= \int \frac{10}{x}\ dx + \int 2025^x\ dx \\ &= 10 \int \frac{1}{x}\ dx + \int 2025^x\ dx \\ &= \boxed{10 \ln|x| + \frac{2025^x}{\ln 2025} + C} \end{aligned}$$

Try another example

Example 5.
Evaluate

$$\int \left(\frac{10^2}{x} - 5^{3x} \right) dx$$

Solution.

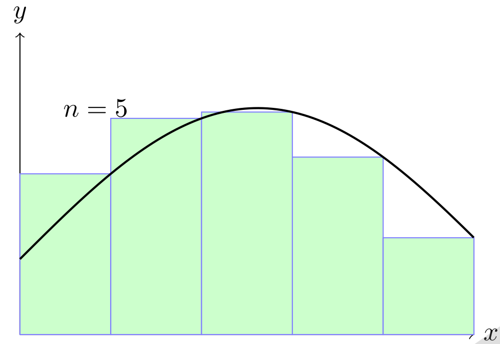
$$\begin{aligned} \int \left(\frac{10^2}{x} - 5^{3x} \right) dx &= 100 \int \frac{1}{x}\ dx - \int 125^x\ dx \\ &= \boxed{100 \ln|x| - \frac{125^x}{\ln 125} + C} \end{aligned}$$

1.4 Definite Integrals

Before this, we only learnt to solve integrals by finding inverse of differentiation and applying some rules. We know that our integrals are always in the form of

$$\int f(x)\ dx = F(x) + C$$

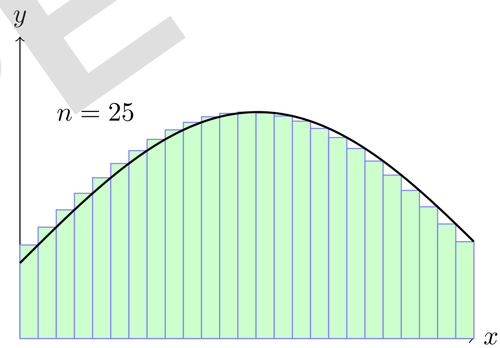
Now, we will learn to visualize this very integral using *Riemann Sums.* Intuitively, integrals essentially find the area under the graph. First, approximate the area using many rectangles underneath the curve of $f(x) = \sin x + 0.5$ over a fixed interval $[a, b]$.



We let the width of each rectangle represent δ_n, and the height as $f(\delta_n)$, where n represents the number of rectangles. Thus, the sum of all rectangles is

$$f(\delta_1) \cdot \delta_1 + f(\delta_2) \cdot \delta_2 + \cdots + f(\delta_5) \cdot \delta_5 = \sum_{n=1}^{5} f(\delta_n) \cdot \delta_n$$

The sum above is obviously a very poor estimation of the area under the graph. However, as we increase n, and decrease the length of δ_n over the same fixed intervals $[a, b]$, the sum of all rectangles starts to resemble the area under the graph



Now, the sum of rectangles is:

$$\sum_{n=1}^{j} f(\delta_n) \cdot \delta_n$$

where $j = 25$
So, as j approaches infinity, δ_n becomes infinitesimal on the same interval, almost approaching zero, causing the sum to be infinitely closer to the actual area under the graph, which is the integral of $f(x)$!

Thus, on the interval $[a, b]$,

$$\int_a^b f(x)\ dx = \lim_{j \to \infty} \sum_{n=1}^{j} f(\delta_n) \cdot \delta_n$$

So now intuitively, integrals are just the sum of very tiny sliver of rectangles. $f(x)$ here refers to the height of a rectangle, and dx represents the width of that rectangle, which is infinitesimal, approaching zero.

Now, I shall introduce two theorems.

Theorem 1.4.1. *If $f(x)$ is continuous on $[a, b]$ and we define a new function*

$$F(x) = \int_a^x f(t)\ dt$$

then:

$$\boxed{F'(x) = f(x)}$$

This theorem essentially explains differentiation and integration are inverse operations.

Theorem 1.4.2. *If $f(x)$ is continuous on $[a, b]$, and $F(x)$ is any antiderivative of $f(x)$ (i.e $F'(x) = f(x)$), then:*

$$\int_a^b f(x)\ dx = \Big[F(x)\Big]_a^b = F(b) - F(a)$$

This theorem tells us how to solve a definite integral. Notice, there is no constant C, as the integral is solved in the given bounds $[a, b]$, resulting in a fixed value.

We can further understand **Theorem 1.4.1** by using **Theorem 1.4.2**.

$$
\begin{aligned}
F(x) &= \int_a^x f(t)\ dt \\
&= \int_a^x t^2\ dt \qquad \text{Using Theorem 1.4.2} \\
&= \left[\frac{t^3}{3}\right]_a^x \\
&= \frac{x^3}{3} - \frac{a^3}{3} \\
F'(x) &= \frac{d}{dx}\left(\frac{x^3 - a^3}{3}\right) \quad \text{a is a constant} \\
&= x^2 \\
&= \boxed{f(x)}
\end{aligned}
$$

As a is a constant, the value of a does not affect the result. Thus, **Theorem 1.4.1** holds, and now you can see precisely why exactly integration and differentiation are opposite operations.

There is another theorem you must know, which can be useful in certain instances.

Theorem 1.4.3. *For any integrable function f on an interval $[a, b]$,*

$$\int_a^b f(x)\ dx = -\int_b^a f(x)\ dx$$

It can be proven using **Theorem 1.4.2**.

Proof. Notice:

$$
\begin{aligned}
\int_a^b f(x)\ dx &= F(b) - F(a) \\
-\int_a^b f(x)\ dx &= F(a) - F(b) \\
&= \int_b^a f(x)\ dx \qquad \text{By using Theorem 1.4.2}
\end{aligned}
$$

□

We must build on certain rules related to **Theorem 1.4.3**.

1. $\int_a^a f(x)\ dx = 0$
2. $\int_a^c f(x)\ dx = \int_a^b f(x)\ dx + \int_b^c f(x)\ dx \qquad \text{For all } a < b < c$

Now, by understanding how definite integrals work, we can attempt to solve these examples below

Example 6. Evaluate

$$\int_6^8 9x^5 - 12\ dx$$

Solution.

$$\begin{aligned}\int_6^8 9x^5 - 12\ dx &= 9\int_6^8 x^5\ dx - \int_6^8 12\ dx \\ &= 9 \cdot \left[\frac{x^6}{6}\right]_6^8 - \left[12x\right]_6^8 \\ &= \left(\frac{3}{2}8^6 - \frac{3}{2}6^6\right) - \left(12(8) - 12(6)\right) \\ &= \boxed{323208}\end{aligned}$$

Example 7. Evaluate

$$\int_{-1}^2 2^{2x}\ dx$$

Solution.

$$\begin{aligned}\int_{-1}^2 2^{2x}\ dx &= \int_{-1}^2 4^x\ dx \\ &= \left[\frac{4^x}{\ln 4}\right]_{-1}^2 \\ &= \frac{1}{\ln 4}\left(4^2 - 4^{-1}\right) \\ &= \boxed{\frac{63}{4\ln 4}}\end{aligned}$$

Now, you can attempt to solve these proof questions to improve your understanding.

Practice 8.

1. Prove: $\int_a^b f(x)\ dx + \int_b^c f(x)\ dx + \int_c^a f(x)\ dx = 0 \qquad \text{where } a < b < c$
2. If $f(x)$ is an odd function, prove: $\int_{-a}^{a} f(x)\ dx = 0$

Chapter 2

Basic Techniques of Integration

2.1 Integration by Substitution

In Chapter 1, we have so far learnt to solve integrals that are usually linear or have known antiderivatives, using very basic rules. However, this limits the range of integrals we can evaluate.
The first technique you will learn is Integration by Substitution. It is commonly known as u-substitution or u-sub.

Theorem 2.1.1. *Let there be two differentiable and continuous functions $f_1(x)$ and $f_2(x)$*

$$\int f_1(f_2(x)) \cdot f_2'(x)\ dx = \int f_1(u)\ du$$

where $u = f_2(x)$

Proof.

$$\int f_1(f_2(x)) \cdot f_2'(x)\ dx$$

Let $u = f_2(x)$. Then you differentiate it:

$$\frac{du}{dx} = f_2'(x)$$

. Then you rearrange it in terms of dx: $dx = \frac{du}{f_2'(x)}$. By u-substitution

$$\int f_1(u) f_2'(x) \cdot \frac{du}{f_2'(x)} = \int \boxed{f_1(u)\ du}$$

□

This theorem is easier to understand through examples.

Example 1.
Evaluate

$$\int (3x^2 + 2)e^{x^3+2x} \, dx$$

Solution.

$$\begin{aligned}
u &= x^3 + 2x \\
du &= (3x^2 + 2) \, dx \\
dx &= \frac{du}{3x^2 + 2} \\
\int (3x^2 + 2)e^{x^3+2x} \, dx &= \int (3x^2 + 2)e^u \frac{du}{3x^2 + 2} \\
&= \int e^u \, du \\
&= e^u + C \\
&= \boxed{e^{x^3+2x} + C}
\end{aligned}$$

u-Substitution is all about smart recognition and understanding. However, it is a little different for definite integrals, as the bounds will change with respect to the substitution made. Take a look at Example 2.

Example 2.
Evaluate
$$\int_1^3 (3x^2+2)e^{x^3+2x}\,dx$$
Leave your answer in terms of e.

Solution. Using the same substitution in Example 1.

$$\int_1^3 (3x^2+2)e^{x^3+2x}\,dx = \int_{u(1)}^{u(3)} (3x^2+2)e^u \frac{du}{3x^2+2}$$
$$= \int_{u(1)}^{u(3)} e^u\,du$$
$$u(1) = 1^3 + 2(1) = 3$$
$$u(3) = 3^3 + 2(3) = 33$$

Thus the bounds change to:

$$= \int_3^{33} e^u\,du$$
$$= e^u\Big|_3^{33}$$
$$= \boxed{e^{33} - e^3}$$

Practice 3
Evaluate

1. $\displaystyle\int \frac{x^3}{x^4+9}\,dx$
2. $\displaystyle\int_2^3 \frac{1}{(2x+5)^3}\,dx$

2.2 Integration by Parts

This technique is often used in many different integrals. This technique is called **Integration By Parts**, which is known to be the inverse of the product rule for differentiation.

Theorem 2.2.1. *Let u and v be differentiable and continuous functions on an interval, then*

$$\int u\ dv = uv - \int v\ du$$

Note: $u = u(x)$ *and* $v = v(x)$

Example 4. Evaluate

$$\int e^x(x+1)\ dx$$

Solution. This can be expanded, but we'll use integration by parts here.
Let $u = x + 1$, so $du = dx$.
Let $dv = e^x\ dx$, so $v = e^x$
Thus,

$$\begin{aligned}\int e^x(x+1)\ dx &= e^x(x+1) - \int e^x\ dx \\ &= e^x(x+1) - e^x + C \\ &= \boxed{xe^x + C}\end{aligned}$$

People struggle with integration by parts, as they do not know which function u should be equal to. There is a special strategy for choosing u when doing integration by parts.
When you have an integral of the form

$$\int u\ dv$$

LIATE is a good guideline for choosing u:
- **L**: Logarithmic Functions ($\ln x$ etc.) - Highest priority
- **I**: Inverse Trigonometric Functions ($\arctan x$ etc.)
- **A**: Algebraic Functions (x^3 etc.)
- **T**: Trigonometric Functions ($\sin x$ etc.)

- **E**: Exponential Functions (e^x etc.) - Lowest priority

Likewise, in **Example 4**, $x+1$ falls under *Algebraic Functions* which has a higher priority of selection for u compared to e^x which falls under *Exponential Functions.*
Using this newfound strategy, you can attempt the following examples with ease.

Example 5.
Evaluate

$$\int_1^2 x\ln x \; dx$$

Leave your answer in terms of $\ln 2$

Solution. First, evaluate the indefinite integral (it is easier to evaluate).
Let $u = \ln x$, so $du = \frac{1}{x}\, dx$
Let $dv = xdx$, so $v = \frac{x^2}{2}$

$$\begin{aligned}\int x\ln x \; dx &= \frac{x^2}{2}\ln x - \int \frac{x^2}{2}\cdot\frac{1}{x}\; dx \\ &= \frac{x^2}{2}\ln x - \frac{1}{2}\int x \; dx \\ &= \frac{x^2}{2}\ln x - \frac{x^2}{4} + C\end{aligned}$$

Now, you can solve with the bounds.

$$\begin{aligned}\int_1^2 x\ln x \; dx &= \left[\frac{x^2}{2}\ln x - \frac{x^2}{4}\right]_1^2 \\ &= (2\ln 2 - 1) - \left(-\frac{1}{4}\right) \\ &= \boxed{2\ln 2 - \frac{3}{4}}\end{aligned}$$

Practice 6
Evaluate

1. $\displaystyle\int \ln x \; dx$

2.3 Integration Using Partial Fractions

Partial Fraction Decomposition is the process of expressing a single rational function as a sum of two or more rational functions, whose denominators are factors of the original denominator. *Formally:*

Theorem 2.3.1.

$$\frac{P(x)}{Q(x)} = \frac{A_1}{f_1(x)} + \frac{A_2}{f_2(x)} + \cdots + \frac{A_n}{f_n(x)}$$

where each $f_i(x)$ is a simpler polynomial factor of $Q(x)$, and the numerators A_i are constants or lower-degree polynomials

Before that, we must understand what a rational function is.

Theorem 2.3.2. *A rational function is a ratio of two polynomials:*

$$R(x) = \frac{P(x)}{Q(x)}$$

*If the degree of P is less than that of Q, then the fraction is called proper; otherwise, you first perform polynomial long division to make it proper, in order to apply **Theorem 2.3.1**.*

Essentially, Partial Fraction Decomposition (PFD) is a precursor to calculus, a topic from pre-algebra. It is recommended for readers to know the partial fraction decomposition well enough to apply it in this section.

Example 7. Solve

$$\int \frac{1}{x(x+1)}\,dx$$

Solution. We must decompose the integrand using Partial Fraction Decomposition.

$$\frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$$

Then, solve

$$1 = A(x+1) + Bx$$

$$A = 1, B = -1$$

$$\begin{aligned}\int \frac{1}{x(x+1)}\,dx &= \int \left(\frac{1}{x} - \frac{1}{x+1}\right) dx \\ &= \int \frac{1}{x}\,dx - \int \frac{1}{x+1}\,dx\end{aligned}$$

$$\text{Let}\quad u = x+1,\ du = dx$$

$$\begin{aligned}\int \frac{1}{x(x+1)}\,dx &= \ln|x| - \ln|x+1| + C \\ &= \boxed{\ln\left|\frac{x}{x+1}\right| + C}\end{aligned}$$

This is actually a known result. See **Chapter 7, General Formulae, Number 16.**.

Try this example. This is a much harder reflection of PFD.

Example 8. Solve

$$\int \frac{3x+1}{(x+1)(x-2)}\,dx$$

Solution. We must decompose the integrand using Partial Fraction Decomposition.

$$\frac{3x+1}{(x+1)(x-2)} = \frac{A}{x+1} + \frac{B}{x-2}$$

Then, solve

$$3x+1 = A(x-2) + B(x+1)$$

$$A = \frac{2}{3}, B = \frac{7}{3}$$

$$\int \frac{3x+1}{(x+1)(x-2)}\,dx = \frac{2}{3}\int \frac{1}{x+1}\,dx + \frac{7}{3}\int \frac{1}{x-2}\,dx$$
$$= \boxed{\frac{2}{3}\ln|x+1| + \frac{7}{3}\ln|x-2| + C}$$

Here is one more example which shows an improper rational fraction, with a quadratic denominator. Recall that all improper fractions have a numerator that has a higher degree than its denominator.

Formally, let

$$J(x) = \frac{i_1(x)}{i_2(x)}$$

a rational function, where $i_1(x)$ and $i_2(x)$ are polynomials. If $\deg(i_1(x)) \geq \deg(i_2(x))$, then $J(x)$ is an improper fraction. $J(x)$ is only proper if $\deg(i_1(x)) < \deg(i_2(x))$.

Improper fractions must be simplified using polynomial long division, such that

$$J(x) = Q(x) + \frac{R(x)}{i_2(x)}$$

where $Q(x)$ is a polynomial and $R(x)$ is the remainder which is a proper rational fraction.

Example 9. Solve

$$\int \frac{x^3 + 2x^2 + 3x + 4}{x^2 + 1}\, dx$$

Solution.
Polynomial long division should be used on the integrand as degree of the numerator is higher than the degree of the denominator, allowing the integrand to now become proper. Then, decomposition of the integrand will take place.

$$\begin{aligned}
\int \frac{x^3 + 2x^2 + 3x + 4}{x^2 + 1}\, dx &= \int (x+2)\, dx + \int \frac{2x+2}{x^2+1}\, dx \\
&= \frac{x^2}{2} + 2x + \int \frac{2x}{x^2+1}\, dx + \int \frac{2}{x^2+1}\, dx \\
&= \boxed{\frac{x^2}{2} + 2x + \ln\left(x^2+1\right) + 2\tan^{-1} x + C}
\end{aligned}$$

The last integral is a known result, and it will be covered in Chapter 3.

There are endless examples and different types of integrals that require partial fractions, and can be solved intuitively.

2.4 Even/Odd Integrals

In definite integrals, certain functions exhibit **symmetry** that allows us to simplify their evaluation quickly.
Recap:

Theorem 2.4.1. *A function $f(x)$ is:*
Even *if $f(-x) = f(x)$*
Odd *if $f(-x) = -f(x)$*

Before proceeding, it is important to note that the derivative of an even function is odd, and the derivative of an odd function is even. Moreover, an antiderivative of an odd function can be chosen to be even, while an antiderivative of an even function can be chosen to be odd (up to an additive constant). Thus, if $f(x)$ is even, then there exists an antiderivative $F(x)$ of $f(x)$ which is odd. I shall introduce you to the following theorems.

Theorem 2.4.2. *Let $f(x)$ be a continuous even function on $[-a, a]$*

Then,

$$f(-x) = f(x)$$

and

$$\int_{-a}^{a} f(x)\ dx = 2\int_{0}^{a} f(x)\ dx$$

Here is the proof of the above theorem.

Proof. Recall $f(x)$ is even. Thus, $F(x)$ is odd.

$$\begin{aligned}\int_{-a}^{a} f(x)\ dx &= F(a) - F(-a) \qquad \text{As } F(-a) = -F(a), \\ &= F(a) - (-F(a)) \\ &= F(a) + F(a) = 2F(a) \\ &= 2(F(a) - F(0)) \\ &= 2\int_{0}^{a} f(x)\ dx\end{aligned}$$

□

Conversely

Theorem 2.4.3. *Let* $f(x)$ *be a continuous odd function on* $[-a, a]$
Then,

$$f(-x) = -f(x)$$

and

$$\int_{-a}^{a} f(x)\ dx = 0$$

Here is the proof of the above theorem.

Proof. Recall $f(x)$ is odd. Thus, $F(x)$ is even.

$$\begin{aligned}\int_{-a}^{a} f(x)\ dx &= F(a) - F(-a) \qquad \text{As } F(-a) = F(a), \\ &= F(a) - F(a) \\ &= \boxed{0}\end{aligned}$$

□

Example 10. Solve

$$\int_{-2}^{2} x^n \, dx$$

when $n = 2$ and $n = 3$

Solution.
When $n = 2$, $f(x) = x^2$ is an even function.
This is because $f(-x) = (-x)^2 = x^2 = f(x)$. Thus,

$$\begin{aligned} \int_{-2}^{2} x^2 \, dx &= 2\int_{0}^{2} x^2 \, dx \\ &= 2\left[\frac{x^3}{3}\right]_0^2 \\ &= \boxed{\frac{16}{3}} \end{aligned}$$

When $n = 3$, $f(x) = x^3$ is an odd function. This is because $f(-x) = (-x)^3 = -x^3 = -f(x)$. Thus,

$$\int_{-2}^{2} x^3 \, dx = \boxed{0}$$

Try this too!

Example 11. Solve,

$$\int_{-\pi}^{\pi} \sin(x) \, dx$$

Solution. $f(x) = \sin x$ is an odd function. Thus, it evaluates to $\boxed{0}$

2.5 King's Property

The King's Property is a very elegant result that can be used to solve integrals that sometimes can be very hard to handle by just using common methods.

Theorem 2.5.1. *If $f(x)$ is continuous on $[a,b]$, then*

$$\int_a^b f(x)\ dx = \int_a^b f(a+b-x)\ dx$$

Here is a proof of the above theorem

Proof. Let $t = a+b-x$
Then, $dt = -dx$
When $x = a$, then $t = b$
When $x = b$, then $t = a$

$$\begin{aligned} \int_a^b f(x)\ dx &= \int_b^a f(a+b-t)(-dt) \\ &= \int_a^b f(a+b-t)\ dt \qquad \text{Replace dummy variable } t \text{ with } x \\ &= \boxed{\int_a^b f(a+b-x)dx} \end{aligned}$$

□

Example 12. Evaluate

$$\int_0^{\frac{\pi}{2}} \frac{\cos^n x}{\sin^n x + \cos^n x}\,dx$$

Solution.

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \frac{\cos^n x}{\sin^n x + \cos^n x}\,dx &= \int_0^{\frac{\pi}{2}} \frac{\cos^n(\frac{\pi}{2}-x)}{\sin^n(\frac{\pi}{2}-x) + \cos^n(\frac{\pi}{2}-x)}\,dx \\ &= \int_0^{\frac{\pi}{2}} \frac{\sin^n x}{\cos^n x + \sin^n x}\,dx\end{aligned}$$

Notice

$$\begin{aligned}2\int_0^{\frac{\pi}{2}} \frac{\cos^n x}{\sin^n x + \cos^n x}\,dx &= \int_0^{\frac{\pi}{2}} \frac{\cos^n x}{\sin^n x + \cos^n x}\,dx \\ &\quad + \int_0^{\frac{\pi}{2}} \frac{\sin^n x}{\cos^n x + \sin^n x}\,dx \\ &= \int_0^{\frac{\pi}{2}} \frac{\cos^n x + \sin^n x}{\sin^n x + \cos^n x}\,dx \\ &= \int_0^{\frac{\pi}{2}} 1\,dx \\ &= \frac{\pi}{2}\end{aligned}$$

Thus

$$\int_0^{\frac{\pi}{2}} \frac{\cos^n x}{\sin^n x + \cos^n x}\,dx = \boxed{\frac{\pi}{4}}$$

Example 13. Evaluate

$$I = \int_0^2 \sqrt{x^2 - x + 1} - \sqrt{x^2 - 3x + 3}\ dx$$

Solution.
Let $f(x) = \sqrt{x^2 - x + 1}$
Using **Theorem 2.5.1**, we observe

$$\int_0^2 f(x)\ dx = \int_0^2 f(2-x)\ dx$$

Thus, we must find $f(2-x)$

$$f(2-x) = \sqrt{(2-x)^2 - (2-x) + 1} = \sqrt{x^2 - 3x + 3}$$

$$\begin{aligned} I &= \int_0^2 \sqrt{x^2 - x + 1}\ dx - \int_0^2 \sqrt{x^2 - 3x + 3}\ dx \\ &= \int_0^2 f(2-x)\ dx - \int_0^2 \sqrt{x^2 - 3x + 3}\ dx \\ &= \int_0^2 \sqrt{x^2 - 3x + 3}\ dx - \int_0^2 \sqrt{x^2 - 3x + 3}\ dx \\ &= \boxed{0} \end{aligned}$$

2.6 Differentiating an integral

This formula helps you differentiate an integral without solving it (this is similar to **Theorem 1.4.1**).

Theorem 2.6.1.

$$\frac{d}{dx}\left(\int_a^x f(t)\ dt\right) = f(x)$$

Now, using this, let us attempt an example.

Example 14.
Let

$$F(x) = \int_0^x \cos(t)\ dt$$

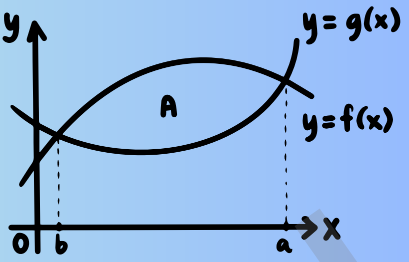
Find $F'(x)$
Solution.

$$\begin{aligned} F'(x) &= \frac{d}{dx}\left(\int_0^x \cos(t)\ dt\right) \\ &= \boxed{\cos(x)} \end{aligned}$$

However, although this does not help us to generally solve integrals, this shows that differentiation and integration are inverse operations. It is important to note that this theorem will be useful in **Chapter 6.3**.

"The Art of Integration" takes the readers on a journey from first principles to advanced techniques. It focuses on building intuition. This book aims to bridge the gap between routine school methods and deeper mathematical understanding. Whether you are encountering integration for the first time or looking to strengthen your mastery, this book helps you see integration as an art.

$$\pi\int_c^d x^2\,dx$$



$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$$

$$\frac{dy}{dx} = \frac{d}{df(x)}g(f(x)) \cdot \frac{d}{dx}f(x)$$

$$\frac{dy}{dx} = nx^{n-1}$$

$$\int_b^a (f(x)-g(x))\,dx$$

AARAV GANDEWAR IS A STUDENT WITH A PASSION FOR MATHEMATICS, AND THIS BOOK REFLECTS HIS EFFORT TO PRESENT MATHEMATICS AS AN ART OF REASONING AND INTUITION

Written by a learner, for learners.

Mathematics is not just solved – it is understood